Discovering Geometry
An Investigative Approach

Tracing Proof in Discovering Geometry

DISCOVERING MATHEMATICS™

Key Curriculum Press
Innovators in Mathematics Education

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Introduction

Discovering Geometry is known for investigations that promote the use of inductive reasoning and cooperative learning, and for connecting geometry to art, architecture, and other disciplines. However, many teachers are less familiar with the development of deductive proof in *Discovering Geometry*. This supplement has been written to help you, the teacher, learn about the approach to proof in *Discovering Geometry* and understand the research that supports it.

For many students, geometry is a significant departure from previous math courses. Although students have seen geometric shapes from their earliest grades, these objects are now being examined, parsed, and compared in discriminating detail. Relatively concrete concepts such as, measurement and position are replaced by more abstract and demanding ideas, like congruence and transformation. The emphasis on visual representation, while refreshing to some students, is a stumbling block for others.

Furthermore, geometry has traditionally been the arena in which logic and deductive thinking are challenged and formalized. Until now, the ideas and language surrounding assumptions, definitions, special cases, counterexamples, and other "players" on the field of mathematical reasoning have taken a back seat to arithmetic, symbol manipulation, sense making, and problem solving. Indeed, research shows that students aren't ready for the abstractions of logic and proof until their teen years. In geometry class, then, we're expected to awaken their deductive faculties, establish processes of proof that will serve them in rigorous debate, and even teach them to appreciate the study of logic as beautiful and quintessential.

Discovering Geometry, in its first edition, was an innovator in addressing students' needs for gradual development of the deductive process, recognizing the importance of the inductive process in conjecture formulation, and utilizing the investigative approach in an area that had been one of the most prescriptive in school mathematics. Old habits die hard, of course, and many who survived "old school" methods and mastered two-column proofs—whether through aptitude or application—have viewed the diminution of emphasis on two-column proof as a dilution of the subject and a "dumbing down" of education.

Discovering Geometry is now in its fourth edition. With each successive edition there has been an examination of its proof strand in the interest of making the strategy more transparent to teachers and the learning process smoother for students. In 1993, Key commissioned proof expert Michael de Villiers to examine the second edition of *Discovering Geometry* and to suggest improvements. The third edition, informed both by de Villiers' review and extensive teacher feedback, featured more deliberate passage from oral explanations to paragraph proof to flow-chart proof to the two-column format. Finally, in this fourth edition, the approach to proof has again been considered and refined. To clarify the proof development for teachers, relevant activities and exercises are flagged with a *Developing Proof* icon or header.

An investigative and open learning process remains uppermost in Key's commitment to students and teachers. We hope this supplement, *Tracing Proof in Discovering Geometry,* will strengthen your conviction in the learning approach, validate the effort you expend in careful development of students' skills, and support your explanations to parents regarding the trajectory of student understanding.

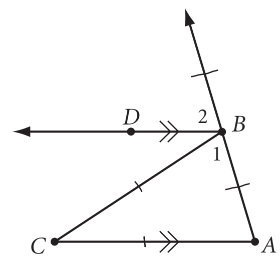
The Development of Proof Skills in *Discovering Geometry*

Experienced geometry teachers realize that many students have trouble learning to write proofs. For example, how many times have you seen a proof like this?

For full credit, your proof must include all steps with justifications and must be in a logical order. Don't forget to mark your diagram.

Given: $\overline{BC} \cong \overline{CA}$ and $\overrightarrow{BD} \parallel \overline{AC}$

Show: $m\angle 1 = m\angle 2$



They do equal the same because they are parrell!

In fact, so many students fail to master two-column proofs that some teachers are skeptical of claims that all students can learn geometry.

Discovering Geometry grew out of the belief that all students *can* learn geometry, including some level of proof. But how? The author, Michael Serra, saw that educational research recommended delaying formal proof until after students had more concrete experience with geometric figures and their properties. Because no geometry books were doing that, he developed his own materials and refined them over years of classroom use.

Discovering Geometry's approach to proof can be summarized in three categories: It uses a developmental approach; it engages students in doing mathematics; and it focuses on proof skills.

A Developmental Approach

Many mathematics teachers have found that students show wide variation in their aptitudes for learning and thinking about geometry, perhaps wider than in other areas of mathematics. The core focus of *Discovering Geometry* helps the majority of students develop a higher level of understanding. You'll also find numerous opportunities to challenge more advanced students and to help students whose progress is slower.

The most widely recognized model for the development of geometric thinking was created by Dina van Hiele-Geldof and Pierre van Hiele (1973). For descriptions in English, see Mayberry (1983), de Villiers (1999), and Battista (2003).

The van Hiele model describes five levels in students' potential for understanding geometric ideas and proof. The model has been supported through independent experimentation and also through practical experience—many geometry teachers

find it describes accurately the different levels of thinking that they observe in their own students. Over the years, limitations in the van Hiele model—for example, that it focuses on how students deal with geometric properties—have been addressed by later researchers. Other aspects of mathematical thinking, such as different roles of proof, are important in the geometry curriculum, so the levels of geometric thinking that guide *Discovering Geometry* extend the van Hiele scheme to include work by others, including Presmeg (1991), Orton (1995), and de Villiers (various). Here is the *Discovering Geometry* model.

- **Level 0** Students think of geometric figures solely as their shapes. A student at Level 0 can distinguish, for example, among triangles, squares, and circles. She might say that she knows a figure is a rectangle because it looks like a door, or that a figure is a circle because it's round. But she is not comfortable discussing properties of figures, such as right angles in a square. Consistent with this focus on general appearance, a student is convinced that a statement is true if it looks true in a picture. You'll recognize Level 0 in work like this.

Tell which figures are parallelograms and which are not parallelograms. Justify your answer.

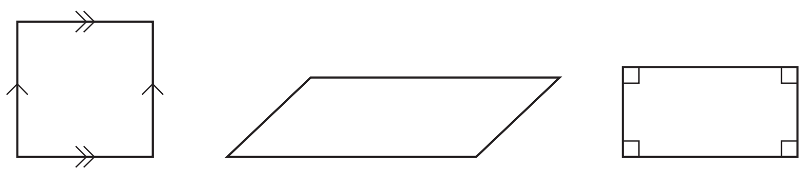


Figure A Figure B Figure C

Figure A is not a parallelogram because it is a square.

Figure B is a parallelogram because it is like a rectangle but slanted.

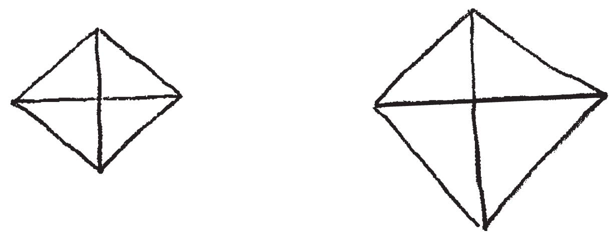
Figure C is a rectangle, not a parallelogram because it is straight.

- **Level 1** Students can discuss the properties of components of figures (for instance, opposite angles of parallelograms are congruent). A student can consider properties when he investigates or makes models. For instance, he can measure to see that isosceles triangles have two congruent angles. On the other hand, he doesn't recognize relationships among properties of figures. For example, he doesn't see that angle congruence in isosceles triangles can be deduced from the congruent sides. Or he might believe that an equilateral triangle can't be isosceles because the list of properties of an equilateral triangle doesn't match the list of properties of an isosceles triangle. Consistent with his reliance on experimentation, a student is convinced that all figures of a type share a property if he can verify the property in a few cases by measuring or model making. That is, he sees no need for verification by means of deductive reasoning. Teachers recognize Level 1 in work like this.

Identify the statement as true or false. Explain your answer.

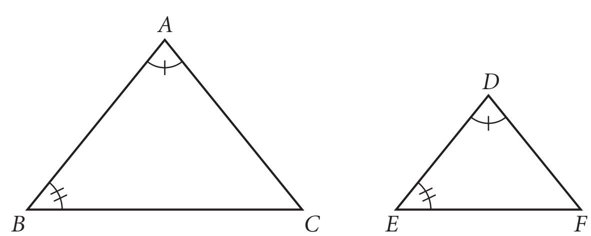
A quadrilateral with diagonals that are perpendicular bisectors of each other is a square.

True. Every time I connected the diagonals, I got a square. I measured the sides and angles to check.



- **Level 2** Students can establish relationships among properties within figures (e.g., in a quadrilateral, if opposite sides are parallel, then opposite angles must be congruent) and within hierarchies of figures (e.g., every square is a rectangle because it has all the properties of a rectangle). A student sees that deductive reasoning can be useful in *explaining* relationships that are not obvious, and she can write informal proofs. But she sees no need to use deduction to *verify* anything, or to include in proofs details that seem obvious. Here's an example of Level 2 work.

Prove that if two angles in a triangle are congruent to two angles in another triangle, then the third angles are also congruent.



Given: $\angle A \cong \angle D$ and $\angle B \cong \angle E$
Show: $\angle C \cong \angle F$

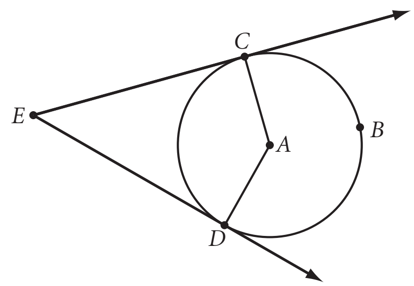
The angles of a triangle are 180°, so $\angle A + \angle B + \angle C = 180°$ and $\angle D + \angle E + \angle F = 180°$. Since $\angle A \cong \angle D$ and $\angle B \cong \angle E$, they must add up to the same thing, so $\angle C$ must be the same as $\angle F$.

- **Level 3** Students realize that what seems obvious may be false. A student can appreciate the use of deduction as a way of verifying for himself the truth of geometric theorems and of justifying his conclusions to others. He understands the interrelationships of undefined terms, postulates, definitions, theorems, and formal proof. He grasps the roles these ideas play in a deductive system. And he can see that very different proofs of the same conclusion can be valid. The work below is an example of Level 3.

Complete the proof of the Intersecting Tangents Conjecture.

Given: $\overrightarrow{EC}$ and $\overrightarrow{ED}$ are tangent to circle A

Show: $m\angle E = 180° - m\widehat{CD}$



Because $\overrightarrow{EC}$ and $\overrightarrow{ED}$ are tangent to circle A, they form right angles with radii $\overline{AC}$ and $\overline{AD}$. So $\angle C$ and $\angle D$ are right angles. ECAD is a quadrilateral, so its angles add up to 360°. So $m\angle E + m\angle C + m\angle A + m\angle D = 360°$. Subtract the right angles: $m\angle E + m\angle A = 180°$. Then solve for $m\angle E$ to get $m\angle E = 180° - m\angle A$. Since $m\angle A$ is a central angle, it has the same measure as $\widehat{CD}$, so substitute to get $m\angle E = 180° - m\widehat{CD}$.

- **Level 4** Students realize that the truth of theorems depends on the deductive system. A student can analyze and compare different deductive systems, such as Euclidean and non-Euclidean geometries. Few high school geometry students attain this level.

Just as you wouldn't expect kindergarteners to move from adding one-digit numbers to combining like terms in an algebraic expression, you should not expect geometry students to jump from Level 0 or 1 directly to Level 4. Most students can't immediately understand the deductive reasoning and logical structures that are so clear to us. Students need to move through each level, and *Discovering Geometry* is designed to provide experiences that will promote this growth.

The van Hieles found that most students begin high school geometry at Level 0 or Level 1, yet most high school geometry courses are taught in a way that assumes students begin at Level 3. The investigations in *Discovering Geometry* begin at Levels 0 and 1 and provide progressive experiences designed to help students move toward Level 3. All along, the text also offers challenges to students who have attained levels higher than the prerequisite level.

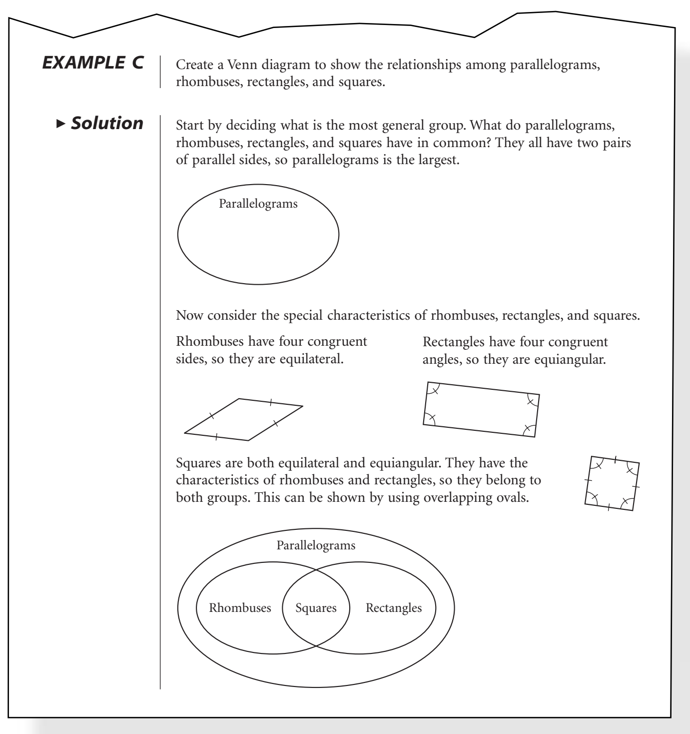
A quick outline shows how *Discovering Geometry* encourages this stepwise maturation in your students' thinking: Chapter 1 generally emphasizes definitions of whole figures and begins to direct students' attention to components of figures. The deductive arguments given for the truth of conjectures in Chapters 2 and 3 are referred to as explanations of the conjectures that arose out of measurements. A mathematician would recognize that these explanations are proofs, but they omit mention of some of the postulates, definitions, or algebraic properties that might distract from the general flow of reasoning. In Chapters 4 through 6, the role of proof as justification comes to the fore. The proofs, presented both in paragraph form and as flowcharts, are more detailed, to help students move from Level 1 to Level 2. Theoretically, the conclusions are still conjectures, because the postulates haven't yet been identified, but all the proofs are mathematically valid.

In Chapters 7 through 12 the focus shifts to the metrics of geometry, while revisiting proof through spiraled review and through examination of the Pythagorean Theorem. This pause allows students time to move through Level 2 and into Level 3. In Chapter 13, for students now at Level 3, the text gives a full set of postulates, including number properties, which can now be used as "reasons" in a two-column proof format. Through the examples and exercises in this chapter, most of the conjectures, as well as some new statements, are classified either as postulates or proved as theorems in full detail.

Differentiating Instruction

Within the developmental approach, you can find challenges for your students who have attained higher levels, as well as foster healthy progress for struggling students. Here are a few examples:

- Much of Chapter 1 is addressed to the goal of helping students at Level 0 define and distinguish components and begin to move to Level 1. But Example C of Lesson 1.9 (page 83) shows a Venn Diagram relating types of special quadrilaterals. Students at Level 0 might only grasp from this example that quadrilaterals are something more than members in a random collection of shapes. They might not comprehend hierarchical relationships. But this example can help students at Level 1 begin to articulate specific ideas about properties within figures, and it can lead to deeper understanding for students at Levels 2 and 3.



EXAMPLE C Create a Venn diagram to show the relationships among parallelograms, rhombuses, rectangles, and squares.

▸ Solution Start by deciding what is the most general group. What do parallelograms, rhombuses, rectangles, and squares have in common? They all have two pairs of parallel sides, so parallelograms is the largest.

Now consider the special characteristics of rhombuses, rectangles, and squares.

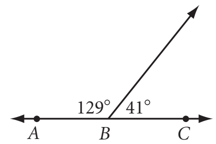
Rhombuses have four congruent sides, so they are equilateral.

Rectangles have four congruent angles, so they are equiangular.

Squares are both equilateral and equiangular. They have the characteristics of rhombuses and rectangles, so they belong to both groups. This can be shown by using overlapping ovals.

- Many *Developing Proof* exercises, such as the one below (page 125), can be approached by students at various levels.

6. ***Developing Proof*** Points *A*, *B*, and *C* at right are collinear. What's wrong with this picture?



A student at Level 0 or 1 might measure the angles and answer the question this way:

The angle on the right is 50° and the angle on the left is 130°. The angle measures are wrong in the diagram.

A student at Level 2 recognizes the relationship of the linear pair and might answer the question like this:

The two angles form a linear pair, so they are supplementary. However, $129° + 41° = 170°$ instead of 180°, so the angle measures cannot be correct.

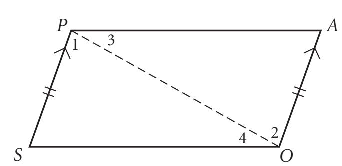
Having the higher level student present her answer to the class can help the students at Level 0 and 1 begin to understand how to use conjectures in their explanations.

- In exercises like the one below (page 301), or in mini-investigations involving proofs, you might find different students giving proofs with varying degrees of detail, depending on their levels. In fact, you can often use the amount of detail a student includes to assess his or her level and to decide how much support or challenge to offer.

4. ***Developing Proof*** Write a flowchart proof to demonstrate that quadrilateral *SOAP* is a parallelogram. ⓗ

Given: Quadrilateral *SOAP* with $\overline{SP} \parallel \overline{OA}$ and $\overline{SP} \cong \overline{OA}$

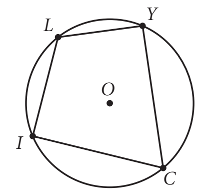
Show: *SOAP* is a parallelogram



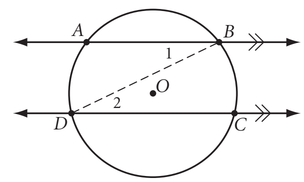
- Lesson 6.4 leads students to prove the Inscribed Angle Conjecture: The measure of an inscribed angle is half that of its intercepted arc. The proof involves three

cases. Students at lower levels might not appreciate the need for this degree of thoroughness, but students at higher levels might relish it. The exercises in this lesson (page 332) provide a number of challenging proofs for higher level students.

3. The opposite angles of a cyclic quadrilateral are supplementary. ⓗ
Given: Circle O with inscribed quadrilateral $LICY$
Show: $\angle L$ and $\angle C$ are supplementary



4. Parallel lines intercept congruent arcs on a circle. ⓗ
Given: Circle O with chord $\overline{BD}$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
Show: $\overset{\frown}{BC} \cong \overset{\frown}{DA}$

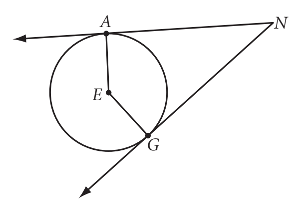


- Starting with Chapter 4, each chapter ends with Take Another Look activities, which extend concepts from the chapter. You might use these as extra credit, as a group challenge to push students to think more deeply, or as part of your regular curriculum for an enriched or honors level course. The activities shown are from Chapter 6 (page 365).

TAKE ANOTHER LOOK

1. ***Developing Proof*** Show how the Tangent Segments Conjecture follows logically from the Tangent Conjecture and the converse of the Angle Bisector Conjecture.

2. ***Developing Proof*** Investigate the quadrilateral formed by two tangent segments to a circle and the two radii to the points of tangency. State a conjecture. Explain why your conjecture is true, based on the properties of radii and tangents.



3. ***Developing Proof*** State the Cyclic Quadrilateral Conjecture in “if-then” form. Then state the converse of the conjecture in “if-then” form. Is the converse also true?

4. ***Developing Proof*** A quadrilateral that *can* be inscribed in a circle is also called a cyclic quadrilateral. Which of these quadrilaterals are always cyclic: parallelograms, kites, isosceles trapezoids, rhombuses, rectangles, or squares? Which ones are never cyclic? Explain why each is or is not always cyclic.

The approach to rectangles in Chapter 5 offers a good example of how students thinking at higher levels are served by the arc of development of a concept. This chapter is geared to Level 2, because it deals with relationships among components of figures and begins to develop hierarchical thinking.

- Lesson 5.3 concentrates on the properties of kites and trapezoids. Although trapezoids are defined to exclude parallelograms, you can encourage your more analytic students to wonder whether the same properties apply to parallelograms.

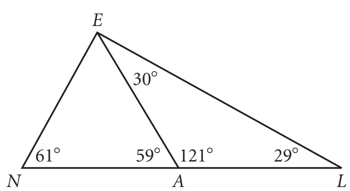
- In Lesson 5.5, which focuses on parallelograms, students measure angles and diagonals of particular parallelograms and make conjectures about parallelograms in general.
- Lesson 5.6 concerns special parallelograms—rhombuses and rectangles. Students, most of whom are at Levels 1 and 2, are asked to repeat the measurements on each type of parallelogram, leading them to discover the hierarchy. You can encourage students at Level 3 to think about the hierarchy from the beginning and to write explanatory proofs based on that understanding.

Traditional approaches tend to abandon students whose progress from level to level is slower than others. If students don't quickly develop the reasoning skills and attention to detail they need to write a two-column proof, they sometimes replace understanding with memorizing, or they give up entirely. In *Discovering Geometry,* students who need more time to attain the next level can gain access to geometric ideas through measurement activities, even if these students grow only into Level 2 by the end of the course. They can learn to write flowcharts as explanations, even if they don't come to see a need for the prerequisite steps and structure that students appreciate at Level 3.

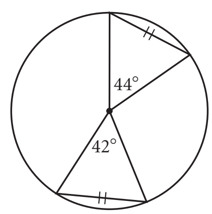
Each lesson includes exercises that provide a spiraled review of previous material. Not only do these review exercises assist slower-developing students, they also give occasion to other students to revisit earlier ideas and relate them to new concepts. The following exercises are from Lesson 4.7 (page 241).

Review

8. ***Developing Proof*** Which segment is the shortest? Explain. ⓗ

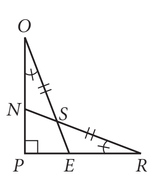


9. ***Developing Proof*** What's wrong with this picture? Explain.

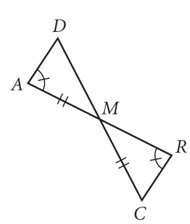


Developing Proof For Exercises 10–12, name the congruent triangles and explain why the triangles are congruent. If you cannot show that they are congruent, write "cannot be determined."

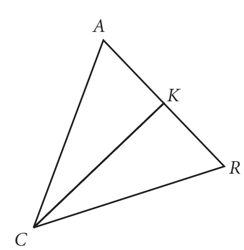
10. $\overline{PO} \cong \overline{PR}$
$\triangle POE \cong \triangle$ _?_
$\triangle SON \cong \triangle$ _?_ ⓗ



11. $\triangle$ _?_ $\cong \triangle$ _?_ ⓗ



12. $\overline{AC} \cong \overline{CR}$, $\overline{CK}$ is a median of $\triangle ARC$. $\triangle RCK \cong \triangle$ _?_



Reasoning Strategies

As mathematics teachers, we have a reserve of problem-solving strategies that we call upon, sometimes unconsciously, when we're faced with a challenging problem. *Discovering Geometry* explicitly develops a set of Reasoning Strategies to help students tackle proofs. Each reasoning strategy is introduced when students first need it to interpret a proof in an example or to write their own in a Developing Proof activity. Then the strategy is reinforced in later examples. Here are the reasoning strategies:

- Draw a labeled diagram and mark what you know.
- Represent a situation algebraically.
- Apply previous conjectures and definitions.
- Break a problem into parts.
- Add an auxiliary line.
- Think backward.

These reasoning strategies are included as a worksheet in *Teaching and Worksheet Masters*. You might consider posting them in your classroom for reference. We'll bring them up again in the upcoming section Aspects of Proof Skills.

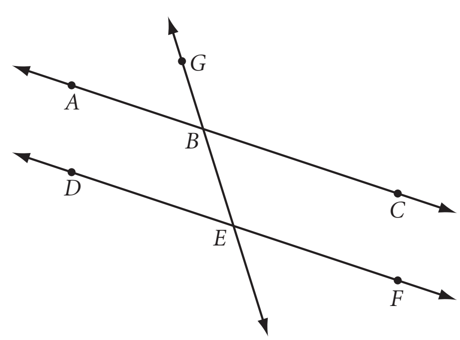
The Practice of Mathematics

Besides teaching the skills of proof developmentally, *Discovering Geometry* helps your students become aware of the importance of proof in the practice of mathematics. Using deductive proof as a developmental or a confirmation stage within deductive systems distinguishes mathematics from other fields. This tradition goes back to Thales of Miletus (ca. 625–547 B.C.E.), one of the earliest Greek mathematicians and philosophers. Thales ("THAY-leez") insisted on logical proof as the way to truth because he saw that results derived from experience could be false.

Often a mathematical investigation begins with *inductive reasoning*, which starts with observing a pattern and leads to making a conjecture. Then the mathematician asks, "But *why* is that conjecture true for all cases?" Enter *deductive reasoning*—looking for a reason, for proof. *Discovering Geometry* is designed so that your students can experience that process.

For instance, suppose that Preethi measures a lot of angles formed by a line intersecting a pair of parallel lines, and she applies inductive reasoning to her observations to make the Corresponding Angles Conjecture (If two parallel lines are cut by a transversal, then corresponding angles are congruent). Then she asks, "But *why*?" So, she draws a diagram to represent the conjecture.

She wants to explain why corresponding angles GBC and GEF are congruent. She might write, "$\angle GEF \cong \angle ABE$ by the Alternate Interior Angles Conjecture, and $\angle ABE \cong \angle GBC$ because they're vertical angles. Therefore, you see that $\angle GEF \cong \angle GBC$."



Note two features of this explanatory proof. First, it depends on the truth of conjectures about vertical angles and alternate interior angles. Are these conjectures in fact true? Can they too be proved? Those questions are not of such

concern to the student in an explanation as they would be to one concerned about verification or systematization. Second, the given proof is only one of many. Perhaps another student might say, "Ah! $\angle GEF$ is just a shift of $\angle GBC$, so of course they're congruent." Again, this explanatory proof depends on an assumption—translation preserves angle measure—that students may or may not consider too obvious to mention.

This example also demonstrates why a deductive argument is preferable to a two-column format in proofs that serve as explanations. Providing a reason for each statement in a two-column proof requires more tying up of loose ends than beginning learners can cope with. For example, if you break down the proof of the Corresponding Angles Conjecture, you not only need to validate the truth of the underlying conjectures, but you must also cite a reason that allows substitution. At this point, most of your students probably won't recognize the need for naming a process as obvious as substitution, much less for justifying it. Indeed, mathematicians use paragraph proofs to report mathematical research because justifying every step would be unwieldy and detract from the explanation; they count on their readers making some of the unwritten connections themselves.

Although the emphasis on the deductive system is postponed until Chapter 13, when students are more ready to appreciate it, it's important that the proofs they see or write in earlier chapters can eventually be fit into such a system. In particular, proofs must not be circular (tautological): You can't assume conjecture A in proving conjecture B, assume the truth of B in proving C, and then assume C in proving A.

Although *Discovering Geometry* does not state the postulates first, the conjectures are proved in a logically valid order, so that when the postulates are articulated in Chapter 13, the remaining conjectures become theorems. (See Appendix A for maps showing this order.) Then, when the structure of the deductive system is discussed in detail, the diagrams showing the "family trees" of central theorems simply shed new light on past work.

Aspects of Proof Skills

Consistent with your students' development and the process of discovering mathematical truths through explanation and justification, *Discovering Geometry* deliberately develops four aspects of proof skills:

- *Properties:* from whole figures to components to relationships to hierarchies
- *Purpose:* from proofs for explaining to proofs for justifying and then to proofs for systematizing
- *Format:* from oral arguments to deductive arguments to paragraph proofs to flowchart proofs to two-column proofs
- *Evidence:* from being convinced by appearance to being convinced by measurement to requiring deductive proof

The table below summarizes how each aspect of proof skill develops sequentially through *Discovering Geometry*.

Chapters	Level	Properties	Purpose	Format	Evidence
0	0	Whole figures			Appearance
1–3	1–2	Components	Explain	Deductive argument	Measurement
4–6	2	Relationships	Explain, justify	Paragraph, flowchart	Measurement, Deduction
7–12	2–3	Relationships, Hierarchies	Justify	Paragraph, flowchart	Deduction
13	3–4	Hierarchies	Systematize	Two-column	Deduction

Throughout the text, many activities and exercises conducive to this development are marked with the notation *Developing Proof.*

To help you weave these four strands together as you teach from *Discovering Geometry,* let's consider each chapter of the text.

The introductory and informal **Chapter 0** considers different kinds of artistic designs. It focuses primarily on the shapes of figures as a whole to provide entrance for students at Level 0, engages students with artistic or visual interests, and gives all students experience using the tools of geometric construction. Students also look at the symmetry of figures, which can be a useful perspective for recognizing components of figures and understanding properties of shapes later in the course.

To advance the *Properties* aspect of proof skill, **Chapter 1** emphasizes moving from whole figures to component parts, while giving students practice with inductive reasoning. Your students are asked to create definitions by looking for patterns in sets of figures. This inductive reasoning requires that they pay attention to component parts of the figures, as does the classify/differentiate model students are coached to use for definitions (page 48). The notion of a counterexample, so important for inductive reasoning, lays the groundwork for deductive reasoning by contradiction. Chapter 1 also helps your students move from being convinced by one example to requiring several examples. In this chapter, the reasoning strategy *Draw a labeled diagram and mark what you know* is introduced.

Beginning Steps to Creating a Good Definition

1. **Classify** your term. What is it? ("A square is a 4-sided figure . . .")
2. **Differentiate** your term. How does it differ from others in that class? (". . . that has four congruent sides and four right angles.")
3. **Test** your definition by looking for a counterexample.

In Chapter 1, students are provided examples of good definitions and also challenged to find counterexamples for incomplete definitions (page 48).

EXAMPLE B

Define these terms:

a. Parallel lines

b. Perpendicular lines

▶ Solution

Following these steps, classify and differentiate each term.

Classify. Differentiate.

a. Parallel lines are lines in the same plane that never meet.

b. Perpendicular lines are lines that meet at 90° angles.

Why do you need to say "in the same plane" for parallel lines, but not for perpendicular lines? Sketch or demonstrate a counterexample to show the following definition is incomplete: "Parallel lines are lines that never meet." (Two lines that do not intersect and are noncoplanar are **skew** lines.)

The first three lessons of **Chapter 2** also focus on inductive reasoning and counterexamples. Then deductive reasoning is introduced for the purpose of explanations, for answering the question *Why?* The conjectures made in this chapter are basics of a deductive system. They concern angles, including those formed by parallel lines and their transversals. Few new definitions are introduced; rather, your students become more familiar with the definitions of Chapter 1 as they use those definitions in developing deductive arguments.

In Lesson 2.4, your students encounter the first of many *Developing Proof* group activities, which are identified by the icon at right. Group activities for developing proofs are valuable for three reasons:

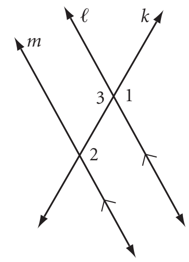
- They appeal to students with oral learning modalities.
- Students' ability to create a clear explanation depends in part on their ability to articulate their own thoughts.
- The activities allow students at different levels to interact. Research in human development (Berkowitz, Gibbs, and Broughton, 1980) indicates that peer discussions may promote more growth than teachers can.

The reasoning strategies introduced in this chapter are *Represent a situation algebraically* and *Apply previous conjectures and definitions.* These strategies are appropriate for all mathematics practitioners, but are especially helpful for students at Level 2, who see proofs as explanations.

At this point, your students may include only reasons in their proofs that they deem necessary, and they won't be concerned with the validity of the reasons. For example, the Linear Pair Conjecture (Angles in a linear pair are supplementary) is cited in proofs without question. Students at Levels 0 and 1 can focus on how it is used to explain other things without being concerned about how it fits into a deductive system. (In Chapter 13, the Linear Pair Conjecture is taken to be a postulate.) At this early stage, it's fine if students don't cite formal reasons for algebraic steps, as shown in the example from Lesson 2.6 (page 131).

EXAMPLE

Write a deductive argument explaining why the Alternate Interior Angles Conjecture is true. Assume that the Vertical Angles Conjecture and Corresponding Angles Conjecture are both true.



▶ ***Solution***

Deductive Argument

In the diagram, lines ℓ and m are parallel and intersected by transversal k. If the Corresponding Angles Conjecture is true, the corresponding angles are congruent.

$\angle 1 \cong \angle 2$

If the Vertical Angles Conjecture is true, the vertical angles are congruent.

$\angle 1 \cong \angle 3$

Because both $\angle 2$ and $\angle 3$ are congruent to $\angle 1$, they're congruent to each other.

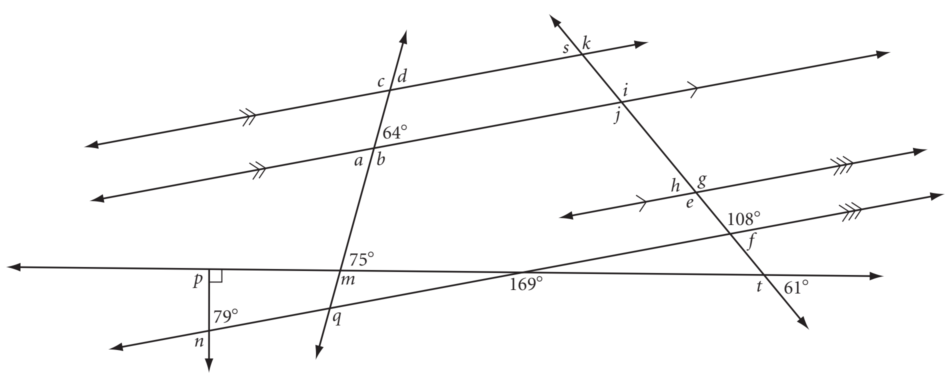
$\angle 2 \cong \angle 3$

Alternate interior angles 2 and 3 are congruent. Therefore, if the corresponding angles are congruent, then the alternate interior angles are congruent. ■

In Chapter 13, however—or earlier in the course for students who have attained higher reasoning levels—you can require that every step in the proof cite a specific property of congruence or equality.

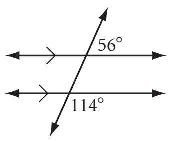
Other activities in Chapter 2 are designed to develop deductive reasoning skills outside the context of a proof. "Angle chases" like the ones from Lesson 2.6, Exercise 7 (page 132), below, help students practice reasoning both forward (*What can I conclude from what I know?*) and backward (*How could I find that angle's measure from what I know?*).

7. ***Developing Proof*** Trace the diagram below. Calculate each lettered angle measure. Explain how you determined measures n, p, and q. ⓗ

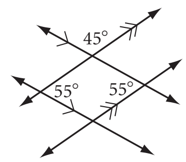


In "What's wrong with this picture" exercises, such as Lesson 2.6, Exercises 9 and 10 (page 133), students begin to deal with reasoning by contradiction. Many students who might be daunted by a request for a proof are engaged by these nonroutine problems.

9. ***Developing Proof*** What's wrong with this picture?

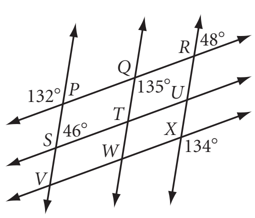


10. ***Developing Proof*** What's wrong with this picture?



"Explain why" and "Explain how you know" exercises, such as Exercise 23 from the Chapter 2 Review (page 142), provide more opportunities for students to develop proof skills. In these exercises, students are doing informal proofs, often using specific measures supplied in the exercise, rather than proving the more abstract general case. This reduces the level of difficulty and lets students focus on the reasoning. It also helps students see proof as another type of problem solving.

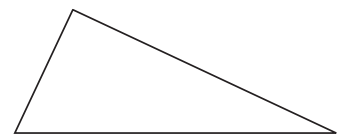
23. ***Developing Proof*** Which pairs of lines are parallel? Explain how you know.



In **Chapter 3,** your students learn to construct geometric figures using patty paper and the geometry tools of straightedge and compass. You will find this chapter valuable for several reasons.

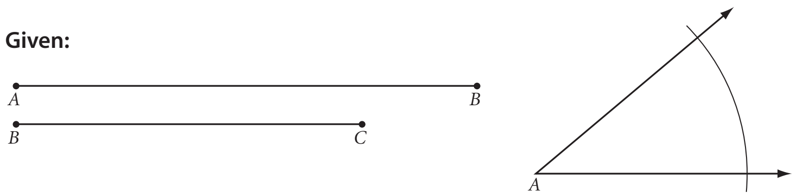
- The construction work helps students who began the course at Level 0 and 1, especially visual and tactile learners, focus more on components of figures.
- Doing constructions gives all students a more kinesthetic understanding of geometric figures. In particular, students gain a physical basis for understanding the triangle congruence shortcuts of Chapter 4 and the properties of quadrilaterals in Chapter 5, so they aren't limited later to memorizing which congruence conditions "work." For example, Exercises 4 and 5 from Lesson 3.6 (page 172), demonstrate that AAA and SSA are not sufficient for determining a triangle. Exercise 5 uses "Given" and "Construct" to parallel the language of "Given" and "Show" used in proofs.

4. Given the triangle shown at right, construct another triangle with angles congruent to the given angles but with sides *not* congruent to the given sides. Is there more than one noncongruent triangle with the same three angles?



5. The two segments and the angle below do not determine a triangle.

Given:



Construct: Two different (noncongruent) triangles named $\triangle ABC$ that have the three given parts ⓗ

- By introducing relatively few new definitions or conjectures, Chapter 3 gives students material to work on while their minds are absorbing the terms and concepts of earlier chapters.
- Constructions raise the level of abstraction somewhat; the focus is no longer on angles with specific measures, for example, but on all possible angles. This is at the core of the *Properties* and *Evidence* aspects of proof skill.
- Some of the discoveries made in this chapter concerning the construction of segments, angles, and parallel and perpendicular lines, are essential underpinnings for the deductive system built up in Chapter 13. That chapter will take as postulates the facts that various figures can be constructed, but those assumptions aren't mentioned now, while most students are unconcerned with a deductive system. The constructions will be cited in less formal proofs before then; in fact, toward the end of Chapter 3, students learn how those constructions are used in proofs of conjectures concerning the incenter and circumcenter.

In Chapter 3, your students are introduced to the reasoning strategy *Break a problem into parts.* This strategy is essential when students begin to work on more complex proofs in later chapters, such as those involving overlapping triangles or relationships between angles and arcs in circles.

While your students investigate triangles in **Chapter 4,** the *Purpose* aspect of proof skill is beginning to mature. Students encounter more relationships that are not quite so apparent, especially with respect to angle measurements in a triangle, so the idea of proof for justification is planted. The proofs are primarily in paragraph form. Students who have attained higher reasoning levels may become aware of how a deductive structure is developing, as more conjectures are proved in terms of the Isosceles Triangle Conjecture (The base angles of an isosceles triangle are congruent) and the triangle congruence conjectures.

After an inductive discovery of the Triangle Sum Conjecture (The sum of the measures of the angles in any triangle is 180 degrees) in Lesson 4.1, student groups are challenged to write paragraph proofs. This assignment (page 201) calls attention to the *Format* aspect of proof skill, readying your students to accept other forms of proof. The text shows how thinking about the "torn angles" of the investigation can lead to the new reasoning strategy *Add an auxiliary line* to a diagram, and then how that diagram in turn can suggest a proof. Students are given questions to discuss

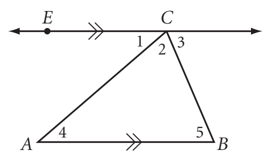
with their group members to help them think through the necessary relationships before writing a proof. Exercises provide follow-up activities.

Developing Proof The investigation may have convinced you that the Triangle Sum Conjecture is true, but can you explain *why* it is true for every triangle?

As a group, explain why the Triangle Sum Conjecture is true by writing a **paragraph proof,** a deductive argument that uses written sentences to support its claims with reasons.

Another reasoning strategy you might use is to add an **auxiliary line,** an extra line or segment that helps with a proof. Your group may have formed an auxiliary line by rearranging the angles in the investigation. If you rotated $\angle A$ and $\angle B$ and left $\angle C$ pointing up, then how is the resulting line related to the original triangle? Draw any $\triangle ABC$ and draw in that auxiliary line.

The figure at right includes $\overleftrightarrow{EC}$, an auxiliary line parallel to side $\overline{AB}$. Use this diagram to discuss these questions with your group.

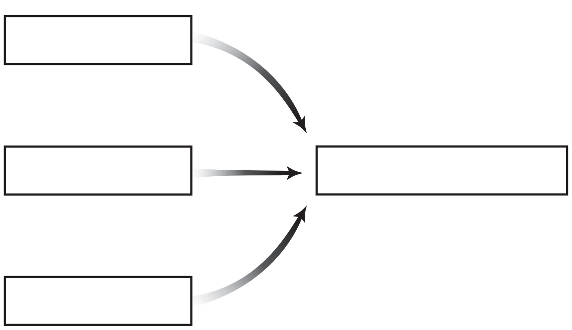


- What are you trying to prove?
- What is the relationship among $\angle 1$, $\angle 2$, and $\angle 3$?
- Why was the auxiliary line drawn to be parallel to one of the sides?
- What other congruencies can you determine from the diagram?

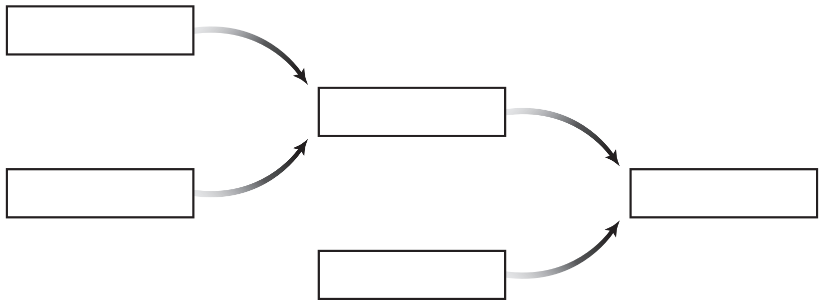
Use your responses to these questions to mark your diagram. Discuss how you can use the information you have to prove that the Triangle Sum Conjecture is true for every triangle. As a group, write a paragraph proof. When you are satisfied with your group's proof, compare it to the one presented on the next page.

In this chapter, proofs become more complex, with justification for one step depending on several previous steps. As an aid to understanding the deductive structure, students begin to use the flowchart form for proofs in Lesson 4.7. The flowchart format appeals to students who learn visually, and it helps students realize that different sequences of statements may be valid in a proof. Some of the flowchart proofs are fill-in-the-blank; students write others from scratch. Your students also gain practice in going back and forth between paragraph and flowchart formats, addressing the *Format* aspect of proof skill.

In the following flowchart, students see that three facts, not in a particular order, contribute to a conclusion.

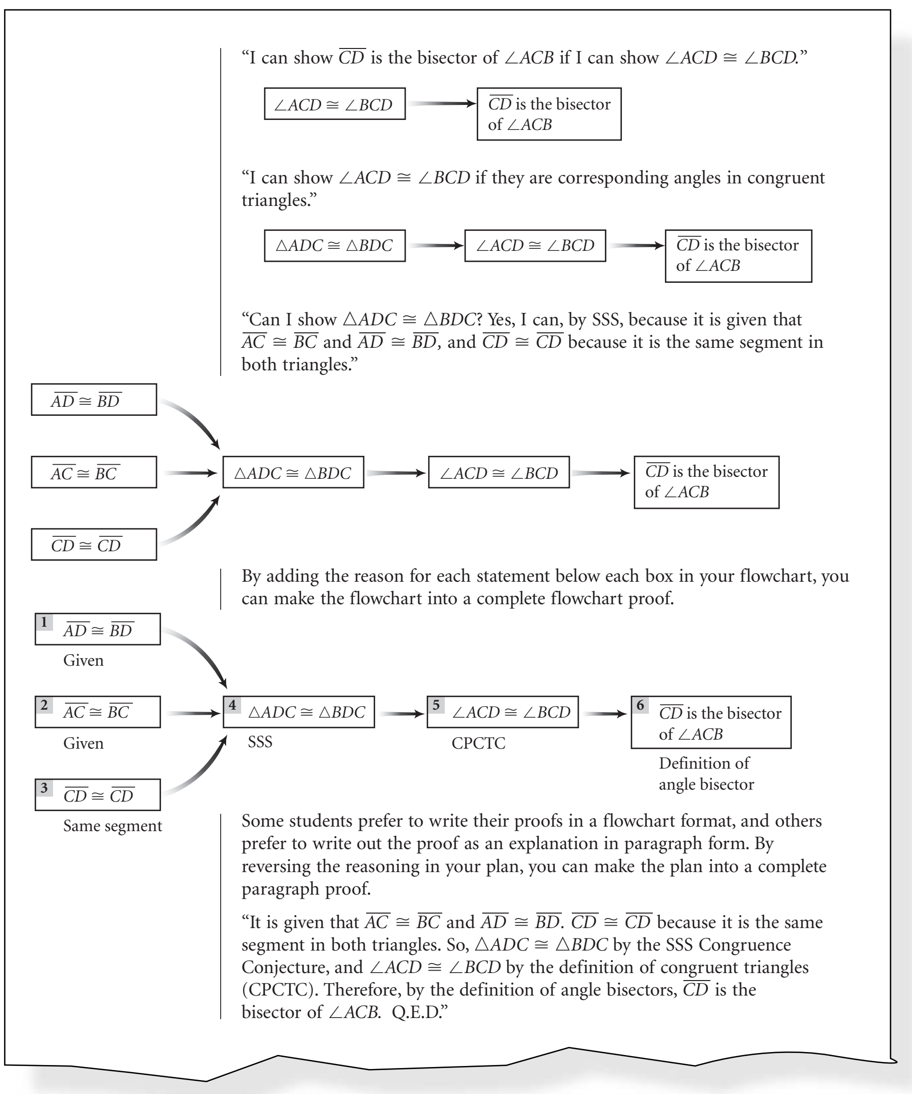


In the next flowchart, students see that the two facts at the far left must be presented before one of the two facts that lead to the result.



Because **Chapter 5** focuses on quadrilaterals, your students get practice relating shapes hierarchically according to their properties. Are all squares parallelograms? Are they trapezoids? Through these kinds of exercises, students further advance in the *Properties* aspect of proof skill.

The flowchart format is useful for the reasoning strategy *Think backward,* which is introduced in Lesson 5.7. Your students can represent visually in a flowchart what's to be proved (in this lesson, that one diagonal of a dart bisects one of the angles) and ask, "How do I prove that?" Working backward from right to left, they ask the same question about each possibility until they find a chain of boxes from the "givens" to the conclusion (page 299).



"I can show $\overline{CD}$ is the bisector of $\angle ACB$ if I can show $\angle ACD \cong \angle BCD$."

$\angle ACD \cong \angle BCD$ → $\overline{CD}$ is the bisector of $\angle ACB$

"I can show $\angle ACD \cong \angle BCD$ if they are corresponding angles in congruent triangles."

$\triangle ADC \cong \triangle BDC$ → $\angle ACD \cong \angle BCD$ → $\overline{CD}$ is the bisector of $\angle ACB$

"Can I show $\triangle ADC \cong \triangle BDC$? Yes, I can, by SSS, because it is given that $\overline{AC} \cong \overline{BC}$ and $\overline{AD} \cong \overline{BD}$, and $\overline{CD} \cong \overline{CD}$ because it is the same segment in both triangles."

$\overline{AD} \cong \overline{BD}$, $\overline{AC} \cong \overline{BC}$, $\overline{CD} \cong \overline{CD}$ → $\triangle ADC \cong \triangle BDC$ → $\angle ACD \cong \angle BCD$ → $\overline{CD}$ is the bisector of $\angle ACB$

By adding the reason for each statement below each box in your flowchart, you can make the flowchart into a complete flowchart proof.

1 $\overline{AD} \cong \overline{BD}$ Given

2 $\overline{AC} \cong \overline{BC}$ Given

3 $\overline{CD} \cong \overline{CD}$ Same segment

4 $\triangle ADC \cong \triangle BDC$ SSS

5 $\angle ACD \cong \angle BCD$ CPCTC

6 $\overline{CD}$ is the bisector of $\angle ACB$ Definition of angle bisector

Some students prefer to write their proofs in a flowchart format, and others prefer to write out the proof as an explanation in paragraph form. By reversing the reasoning in your plan, you can make the plan into a complete paragraph proof.

"It is given that $\overline{AC} \cong \overline{BC}$ and $\overline{AD} \cong \overline{BD}$. $\overline{CD} \cong \overline{CD}$ because it is the same segment in both triangles. So, $\triangle ADC \cong \triangle BDC$ by the SSS Congruence Conjecture, and $\angle ACD \cong \angle BCD$ by the definition of congruent triangles (CPCTC). Therefore, by the definition of angle bisectors, $\overline{CD}$ is the bisector of $\angle ACB$. Q.E.D."

Chapter 6 focuses on circles. Virtually every conjecture is proved—in the text or exercises—in terms of previous conjectures. Most given proofs are in flowchart format, though for exercises students may choose their formats. All three cases of the Inscribed Angle Conjecture (The measure of an angle inscribed in a circle is one-half the measure of the central angle) are proved in Lesson 6.4, thus giving your students an awareness of the careful thinking they need for a rigorous proof.

By Chapter 6, the text has guided most students to move into Level 2 (while encouraging students at higher levels to move toward Level 3). Growth does not always happen in a linear fashion, however. Your students need time to mature. **Chapters 7–12** help students become accustomed to Level 2, while laying some groundwork for the Level 3 approach of Chapter 13. For example, these chapters firmly support the *Purpose* aspect of proof as justification, and they advance the *Evidence* aspect by emphasizing that proofs, not merely examples, are needed for certainty. The format continues to include both paragraph and flowchart proofs.

Chapter 7 focuses on tessellations and includes a proof that all triangles tessellate the plane. The emphasis of **Chapter 8** is on areas. The text is still not explicitly stating "obvious" reasons (such as that congruent figures have the same area), but students derive valid proofs through investigations in which they discover how to rearrange parts of shapes to make other shapes. **Chapter 9** builds on previous work to prove the Pythagorean Theorem in several ways, and it builds on the Pythagorean Theorem to prove other results about right triangles. Students investigate special right triangles inductively and then verify the truth of their conjectures through deductive proofs.

Chapter 10, on volumes, does not include many proofs, because the needed mathematical foundation of metric spaces and calculus isn't part of high school geometry courses. Most conjectures about the similarity results stated in **Chapter 11** are not proved, but you can use the question of how they *might* be proved to motivate the more rigorous treatment in Chapter 13, after the AA Similarity Conjecture is taken to be a postulate. **Chapter 12,** on trigonometry, is concerned primarily with definitions, though proofs are given of the three major theorems. **Chapters 10–12** also offer explorations of symbolic logic, in which students explore types of valid reasoning and proofs, including indirect proof (page 680).

Indirect Proof

Premises: $R \to S$

$\sim R \to \sim P$

P

Conclusion: S

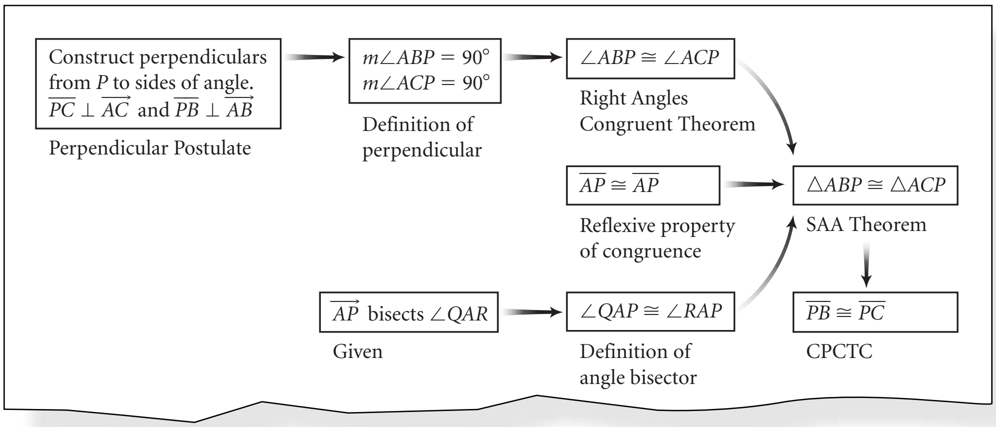
1. $\sim S$	**1.** Assume the opposite of the conclusion
2. $R \to S$	**2.** Premise
3. $\sim R$	**3.** From lines 1 and 2, using MT
4. $\sim R \to \sim P$	**4.** Premise
5. $\sim P$	**5.** From lines 3 and 4, using MP
6. P	**6.** Premise

But lines 5 and 6 contradict each other. It's impossible for both P and $\sim P$ to be true.

Therefore, $\sim S$, the original assumption, is false. If $\sim S$ is false, then S is true.

$\therefore S$

Chapter 13 presents the idea of a deductive system for students who have arrived at Level 3. The *Purpose* aspect of proof moves from justification to systematization. The two-column proof format is introduced, now that attention is being paid to all the pieces of the system that might serve as reasons for statements in a proof. Your students see the basic properties of arithmetic and equality and the postulates of this standard deductive system. All of the early conjectures, and many others, are either stated as postulates or proved within the system, showing all details. "Logical family trees" are used to trace the development of various theorems. The transition to two-column proofs is supported by showing the same proof in flowchart and two-column form, and by using arrows in the two-column proof to demonstrate the logical flow (page 711).



Here is the same proof from the example above, following the same plan, presented as a two-column proof. Arrows link the steps.

Statement	Reason
1. $\overrightarrow{AP}$ bisects $\angle QAR$	**1.** Given
2. $\angle QAP \cong \angle RAP$	**2.** Definition of angle bisector
3. $\overline{AP} \cong \overline{AP}$	**3.** Reflexive property of congruence
4. Construct perpendiculars from P to sides of angle so that $\overline{PC} \perp \overrightarrow{AC}$ and $\overline{PB} \perp \overrightarrow{AB}$	**4.** Perpendicular Postulate
5. $m\angle ABP = 90°$, $m\angle ACP = 90°$	**5.** Definition of perpendicular
6. $\angle ABP \cong \angle ACP$	**6.** Right Angles Congruent Theorem
7. $\triangle ABP \cong \triangle ACP$	**7.** SAA Theorem
8. $\overline{PB} \cong \overline{PC}$	**8.** CPCTC

Not all students will attain Level 3 and appreciate how rigorous proofs fit into the context of a deductive system. As many geometry teachers have discovered, however, more students will have the opportunity to reach Level 3 if you use the developmental approach outlined here, rather than confronting them with formal proofs without sufficient preparation. And even those students who can't fully grasp the material of Chapter 13 will have learned the concepts of geometry through hands-on investigations, come to understand the interplay of inductive and deductive reasoning in doing mathematics, and developed a Level 2 perspective on geometry.

Summary

Discovering Geometry promotes student success with proof in several ways:

- It uses a research-based developmental approach. It builds proof-writing skills gradually through exercises in deductive reasoning, the identification of reasoning strategies, group proof activities, and carefully sequenced experiences writing proofs.
- It provides challenges for students at higher developmental levels as well as opportunities for struggling students to experience success through more concrete experiences.
- It engages students in doing mathematics, moving back and forth between inductive exploration of specific cases and deductive reasoning about general cases.
- It gradually develops proof skills in four areas: attention to properties of figures, complexity of proof format, the purpose of proof, and what constitutes evidence of certainty.

Discovering Geometry leads your students from experience with geometric ideas, through conjectures, to proofs of those conjectures. We are confident that this approach will help your students better understand and appreciate proof.

Assessing Student Work

Proof Rubric

Rather than assigning a number of points to each "statement" and "reason" in a proof, consider using a rubric to assess proofs. Your rubric may change as students become more proficient at proofs and as they move from proofs as explanation to proofs for justification. In early proofs, a diagram is provided in the text, so you might require that students copy and mark the diagram. For later proofs, students may need to create their own diagram that reflects the given information.

Here is a sample assessment rubric.

5-POINT RUBRIC

5 points Given information is clearly stated. The diagram is labeled and marked correctly. The proof is clear and correct with all statements supported by reasons.

4 points Given information is stated. The diagram is labeled and marked. Some markings may be missing or incorrect. The proof is correct, perhaps with a few missing reasons or some redundancy.

3 points The given information is stated, but the diagram is not marked or not correctly marked. The proof contains some correct steps with reasons and the correct conclusion but is missing one or more significant intermediate steps. Or, the proof is incomplete but has sound logic in the steps that are shown.

2 points The given information is not complete. A diagram is incomplete or incorrectly marked. Some true statements are given without justification.

1 point The proof is largely incomplete. Statements are false or unrelated to the conclusion.

Sample Student Work with Commentary

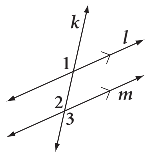
BEGINNING PROOFS: CHAPTER 2

In Lesson 2.6, the example shows a proof of the Alternate Interior Angles Conjecture using the Vertical Angles Conjecture and Corresponding Angles Conjecture. This exercise (page 132) follows up on the example.

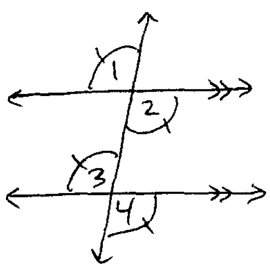
8. ***Developing Proof*** Write a deductive argument explaining why the Alternate Exterior Angles Conjecture is true. Assume that the Vertical Angles Conjecture and Corresponding Angles Conjecture are both true.

The solution from the *Solutions Manual* is shown below.

8. In the diagram, lines l and m are parallel and intersected by transversal k. By the Corresponding Angles Conjecture, $m\angle 1 = m\angle 2$. By the Vertical Angles Conjecture, $m\angle 2 = m\angle 3$. Substitute $m\angle 1$ for $m\angle 2$ in the last equation to get $m\angle 1 = m\angle 3$. The alternate exterior angles 1 and 3 are congruent. Therefore, if two parallel lines are cut by a transversal, then alternate exterior angles are congruent. (This is the Alternate Exterior Angles Conjecture.)



Here is a sample student response. How would you grade it using the 5-point rubric?



∠1 and ∠2 are congruent because of Vertical Angles are congruent. And ∠3 ≅ ∠4 because of Vertical Angles are congruent. So ∠1 ≅ ∠4 because they are all congruent.

The student has correctly drawn a diagram and marked both parallel lines and congruent angles. His proof is missing the step of establishing one more pair of congruent angles through the Corresponding Angles Conjecture: either $\angle 1$ and $\angle 3$, or $\angle 2$ and $\angle 4$. Because the student marked all the angles to be congruent, he may not have realized that this step is missing from the written proof. Because this proof is missing a significant step, this proof merits 3 points.

FOCUS ON PROOF: CHAPTERS 4 TO 6

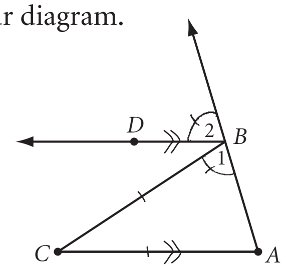
In Chapters 4 through 6, students gain experience writing proofs about triangles, quadrilaterals, and circles. You'll probably adjust your expectations as students become more fluent in the language of proofs. Especially in Chapter 4, do not expect students to justify every step. When students are introduced to flowchart proofs in Chapter 5, they may become more aware of the need to justify each step.

The following proof might be used in a cumulative midyear exam. Several examples of student work along with an analysis and score on the 5-point rubric are shown.

For full credit, your proof must include all steps with justifications and must be in a logical order. Dont fo rget to mark your diagram.

Given: $\overline{BC} \cong \overline{CA}$ and $\overrightarrow{BD} \parallel \overline{AC}$

Show: $m\angle 1 = m\angle 2$



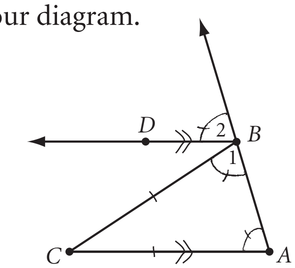
if $\triangle ABC$ is isosceles with $\overline{BC} \cong \overline{CA}$ then the ~~sides~~ angles opposite those sides are congruent, so that means $\overline{BC} \cong \overline{CA} \rightarrow \angle 1 \cong \angle 2$.

This student correctly states the Isosceles Triangle Theorem but incorrectly applies it. There is no congruence mark at $\angle A$ to indicate that the student understands which angles in the triangle are opposite the congruent sides. This proof merits 2 points.

For full credit, your proof must include all steps with justifications and must be in a logical order. Don't forget to mark your diagram.

Given: $\overline{BC} \cong \overline{CA}$ and $\overrightarrow{BD} \parallel \overline{AC}$

Show: $m\angle 1 = m\angle 2$



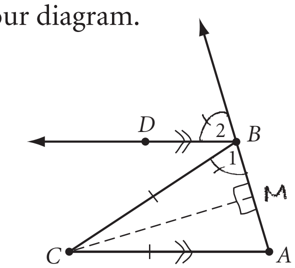
$\triangle ABC$ is isosceles $\triangle$ so $\angle A \cong \angle 1$.
$\angle 2 \cong \angle A$ because they are corresponding angles. Therefore $\angle 1 \cong \angle 2$.

This student correctly used the Isosceles Triangle Conjecture but did not state it as justification. The student correctly uses the Corresponding Angles Conjecture but does not mention the parallel lines and then uses transitivity without justification. This proof would earn 4 points.

For full credit, your proof must include all steps with justifications and must be in a logical order. Don't forget to mark your diagram.

Given: $\overline{BC} \cong \overline{CA}$ and $\overrightarrow{BD} \parallel \overline{AC}$

Show: $m\angle 1 = m\angle 2$



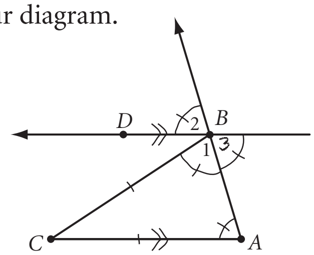
statement	reason
BC ≅ CA	given
BD ∥ AC	given
∠ ACM ≅ ∠BCM	bisect
m∠1 = m∠2	cpcts

This student gets off track and does not make any significant headway towards a proof. The proof would earn 1 point.

For full credit, your proof must include all steps with justifications and must be in a logical order. Don't forget to mark your diagram.

Given: $\overline{BC} \cong \overline{CA}$ and $\overrightarrow{BD} \parallel \overline{AC}$

Show: $m\angle 1 = m\angle 2$



$\boxed{\overline{BC} \cong \overline{CA}}$ → $\boxed{\angle 1 \cong \angle A}$

given

isosceles △ → ∡'s ≅

$\boxed{\overrightarrow{BD} \parallel \overline{AC}}$ → $\boxed{\angle A \cong \angle 3}$

given

Alt. Int's Angles ≅

$\boxed{\angle 1 \cong \angle 3}$

Same as ∠A

$\boxed{\angle 2 \cong \angle 3}$

Vertical ∡'s

$\boxed{\angle 1 \cong \angle 2}$

Same as ∠3

This proof is complete and correct. The student correctly states all reasons except for the transitive property, and the diagram is marked, with an additional label added where needed. This proof would merit 5 points.

PROOF MASTERY: CHAPTER 13

By Chapter 13, you can assume a higher standard for proofs. Students should provide reasons for all statements. Proofs are longer and thus flowcharts are sometimes unwieldy; students may prefer two-column proofs for this reason. Students are often given only a statement of the theorem and they must create a diagram and the Given and Show statements, as well as an initial plan for their proof. This is Exercise 3 from Lesson 13.4 (p. 717).

3. Each diagonal of a rhombus bisects two opposite angles. (Rhombus Angles Theorem)

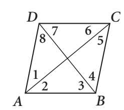
The solution in the *Solutions Manual* shows a sample diagram, the given and show statements, a plan for completing the proof, and a complete two-column proof.

3. Given: Rhombus

Show: Each diagonal of the rhombus bisects two opposite angles

Given: Rhombus $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$

Show: $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$; $\overline{BD}$ bisects $\angle ADC$ and $\angle CBA$



Plan: Use the definition of rhombus, the reflexive property, and the SSS Congruence Postulate to get $\triangle ABC \cong \triangle ADC$. Then use CPCTC and the definition of angle bisector to prove that $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$. Repeat using diagonal $\overline{BD}$.

Proof:

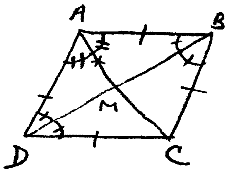
Statement	Reason
1. $ABCD$ is a rhombus	**1.** Given
2. $\overline{AB} \cong \overline{AD}$	**2.** Definition of rhombus
3. $\overline{BC} \cong \overline{DC}$	**3.** Definition of rhombus
4. $\overline{AC} \cong \overline{AC}$	**4.** Reflexive property of congruence
5. $\triangle ABC \cong \triangle ADC$	**5.** SSS Congruence Postulate
6. $\angle 1 \cong \angle 2$; $\angle 6 \cong \angle 5$	**6.** CPCTC
7. $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$	**7.** Definition of angle bisector
8. $\overline{AD} \cong \overline{CD}$	**8.** Definition of rhombus
9. $\overline{AB} \cong \overline{CB}$	**9.** Definition of rhombus
10. $\overline{BD} \cong \overline{BD}$	**10.** Reflexive property of congruence
11. $\triangle ADB \cong \triangle CDB$	**11.** SSS Congruence Postulate
12. $\angle 3 \cong \angle 4$; $\angle 8 \cong \angle 7$	**12.** CPCTC
13. $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$	**13.** Definition of angle bisector

A possible student solution is shown on the next page.

Given: ABCD is a rhombus

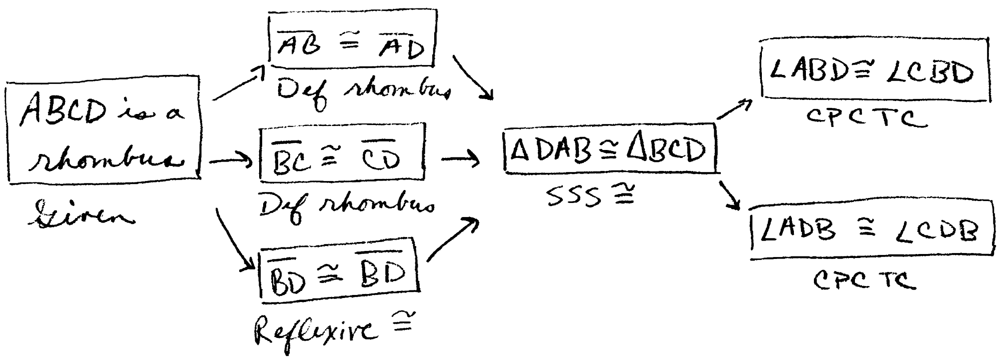
Show: $\overline{AC}$ bisects $\angle A$ and $\angle C$

$\overline{BD}$ bisects $\angle B$ and $\angle D$



Plan: Show that $\triangle DAB \cong \triangle BCD$ and $\triangle ABC \cong \triangle CDA$

Use CPCTC to get $\cong$ $\angle$'s. Same for other $\triangle$'s + $\angle$'s.



Do the same thing for the other triangles.

This student correctly makes the Given and Show statements, draws a diagram marking some of the congruent parts, and has a reasonable plan for the proof although the triangles to be proved congruent do not have the correct correspondence. The proof is partially complete and has some errors. For example, the first two pairs of congruent sides are not corresponding sides for the triangles named in the triangle congruence statement. The triangles proved congruent do not have the corresponding parts cited in the CPCTC statements. Additionally, the student does not tie the congruent angles to the ultimate goal, to show that the diagonal bisects the angles. Finally, the statement of how to complete the other half of the proof is vague; it does not name the triangles that will be used, the method for proving them congruent, or which parts will be shown to be congruent by CPCTC. This proof would merit 3 points on the 5-point scale.

APPENDIX A Proof Maps

The following proof maps are a visual representation of the structure and organization of the conjectures of *Discovering Geometry*. Geometric relationships are first explored through experimentation and then stated as conjectures. Many of the conjectures from Chapters 2 through 12 are proved locally using only previously accepted conjectures and established definitions. In Chapter 13, students build a deductive system from the ground up, beginning with a set of assumptions: undefined terms, definitions, properties of arithmetic and equality, and geometry postulates—which consist of Euclid's postulates and some of the core conjectures from earlier chapters. They write rigorous proofs using those assumptions as reasons, establishing many of the conjectures from Chapters 2 through 12 as theorems within the system. After a conjecture has been established as a theorem, it can be used in subsequent proofs.

In many cases, students may see or write a proof when the conjecture is stated, and then later be asked to prove it again within the deductive system. For example, the Isosceles Triangle Conjecture (If a triangle is isosceles, then the base angles are congruent) is discovered inductively in Lesson 4.2, proved deductively in Lesson 4.7 (shown in Proof Map C), and established as a theorem in the deductive system in Lesson 13.1. Only the first proof of each conjecture is detailed in a proof map.

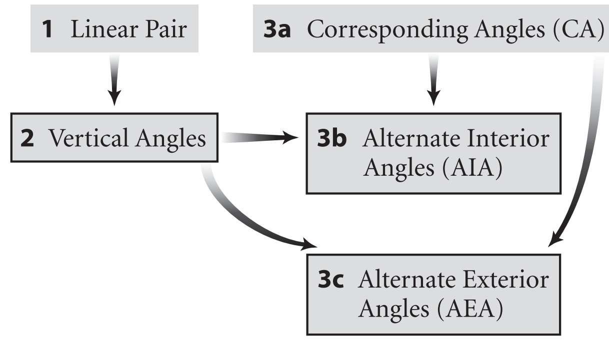
Some conjectures are not proved deductively until Chapter 13, when students have all of the tools and experience they need to complete more complex proofs. This is true of many of the similarity conjectures, which are established as theorems in Lesson 13.7 (see Proof Maps O and P). Because Proof Maps M through P represent Chapter 13 proofs, they cite postulates and theorems rather than conjectures.

Some conjectures are not proved deductively because the mathematics involved is beyond the scope of a high school geometry course. This is true of most of the conjectures from Chapter 7 (about transformations and tessellations) and Chapter 10 (about surface area and volume).

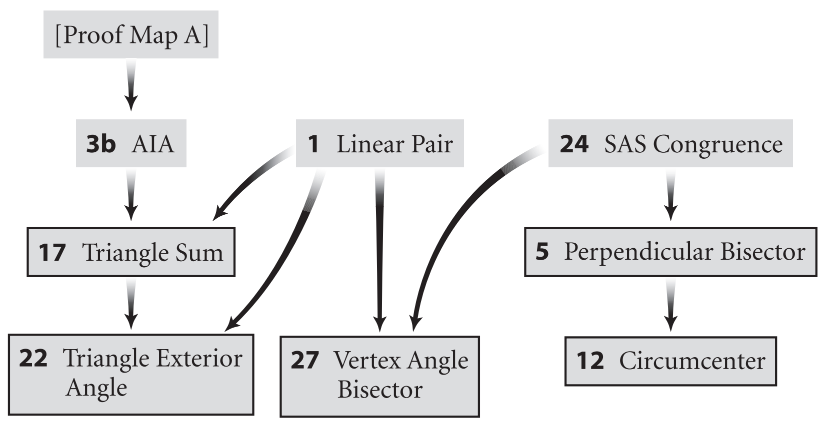
The following index lists the theorems by their original conjecture numbers and shows the proof maps in which they are referenced. If the conjecture is proved within a map, then the map letter is bold in the index. In the map, this is indicated by a border around the conjecture box. Definitions and properties of algebra and arithmetic are not included in these maps.

1 Linear Pair (A, B, D, E, K)
2 Vertical Angles (**A**)
3a Corresponding Angles (A, F, O, P)
3b Alternate Interior Angles (**A**, B, F, I)
3c Alternate Exterior Angles (**A**)
5 Perpendicular Bisector (**B, M**)
6 Converse of the Perpendicular Bisector (**M**)
8 Angle Bisector (**N**)
9 Angle Bisector Concurrency (**N**)
10 Perpendicular Bisector Concurrency (**M**)
12 Circumcenter (**B**)
13 Incenter (**N**)
17 Triangle Sum (**B**, C, D, K, L)
18 Isosceles Triangle (**C**, F, **H**, I)
19 Converse of the Isosceles Triangle (**C**)
22 Triangle Exterior Angle (**B**, I)
23 SSS Congruence (C, E, H, K, M, O)
24 SAS Congruence (B, E, G, M, O)
25 ASA Congruence (C, F, N, P)
26 SAA Congruence (**C**, H, L, N, P)
27 Vertex Angle Bisector (**B**)
28 Equilateral/Equiangular Triangle (**C**, L)
29 Quadrilateral Sum (**D**)
30 Pentagon Sum (**D**)
31 Polygon Sum (D)
32 Exterior Angle Sum (**D**)
33 Equiangular Polygon (**D**)
36 Kite Diagonal Bisector (**E**, J)
37 Kite Angle Bisector (**E**)
39 Isosceles Trapezoid (**F**, G)
40 Isosceles Trapezoid Diagonals (**G**)
42 Triangle Midsegment (**P**)
43 Trapezoid Midsegment (**P**)
44 Parallelogram Opposite Angles (**F**)
46 Parallelogram Opposite Sides (**F**)
47 Parallelogram Diagonals (**F**, G)
49 Rhombus Diagonals (**E**, G)
50 Rhombus Angles (**E**)
51 Rectangle Diagonals (**G**)
52 Square Diagonals (**G**)
53 Tangent (H)
54 Tangent Segments (**H**)
55 Chord Central Angles (**H**)
56 Chord Arcs (**H**)
57 Perpendicular to a Chord (**H**)
58 Chord Distance to Center (**H**)
60 Inscribed Angle (**I**)
61 Inscribed Angles Intercepting Arcs (**I**)
62 Angles Inscribed in a Semicircle (**I**)
63 Cyclic Quadrilateral (**I**)
64 Parallel Lines Intercepted Arcs (**I**)
72 Tessellating Triangles (**D**)
74 Rectangle Area (J, K)
75 Parallelogram Area (**J**)
76 Triangle Area (**J**, K)
77 Trapezoid Area (**J**)
78 Kite Area (**J**)
79 Regular Polygon Area (**J**)
81 The Pythagorean Theorem (**K**, L)
82 Converse of Pythagorean Theorem (**K**)
83 Isosceles Right Triangle (**L**)
84 30°-60°-90° Triangle (**L**)
85 Distance Formula (**L**)
91 AA Similarity (O, P)
92 SSS Similarity (**O**)
93 SAS Similarity (**O**, P)
98 Parallel/Proportionality (**P**)
100 SAS Triangle Area (**J**)
102 Law of Cosines (**L**)

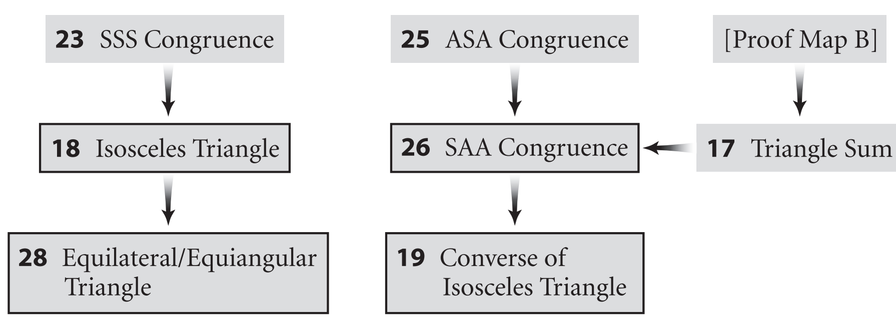
Proof Map A



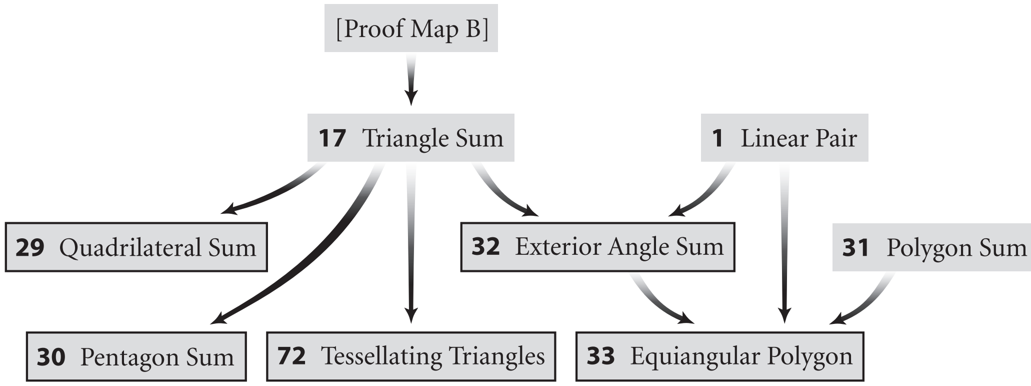
Proof Map B



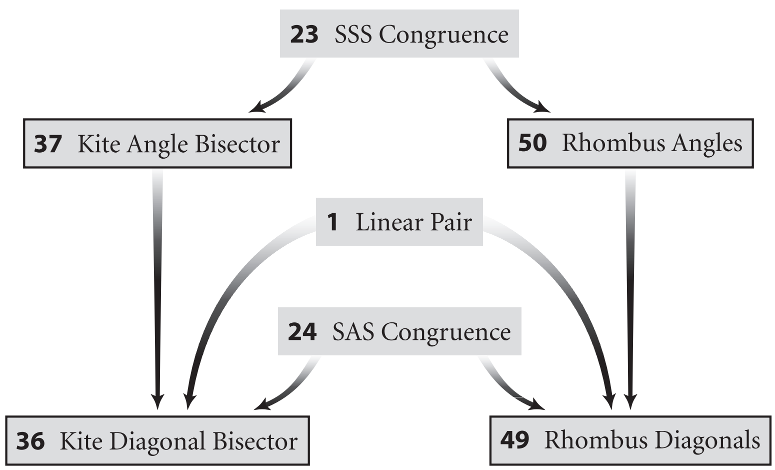
Proof Map C



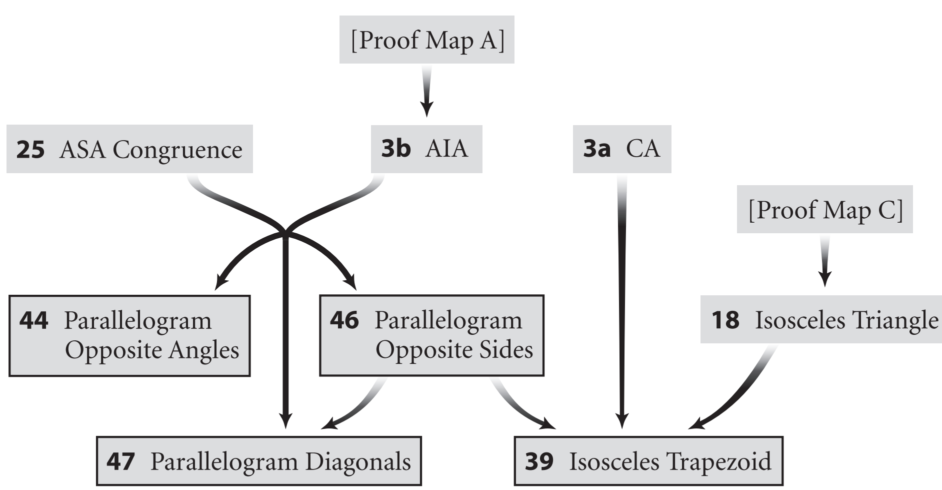
Proof Map D



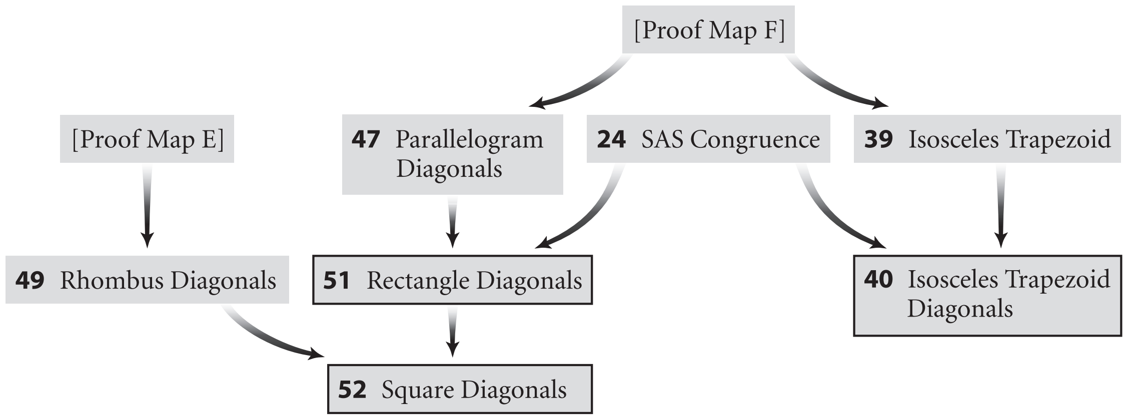
Proof Map E



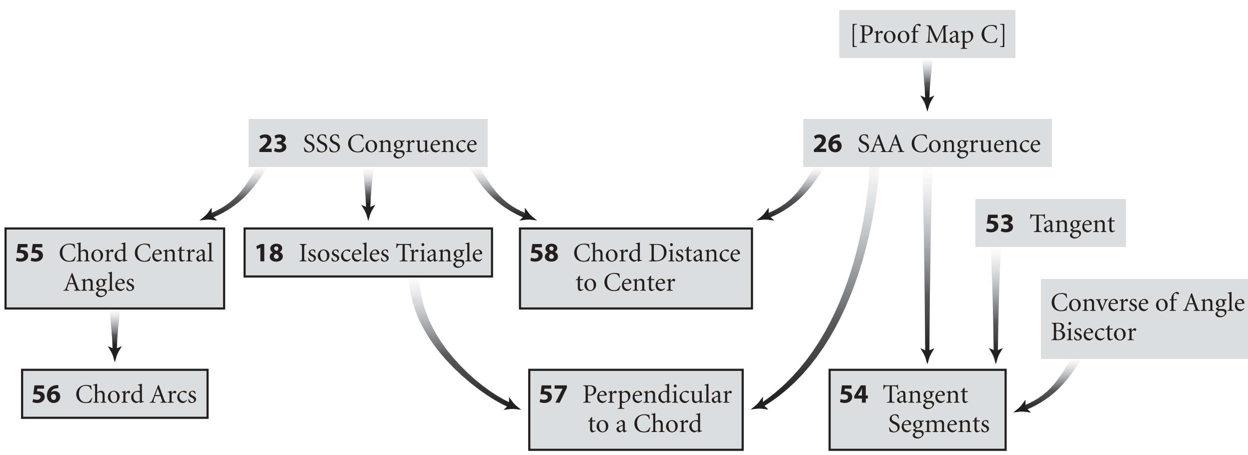
Proof Map F



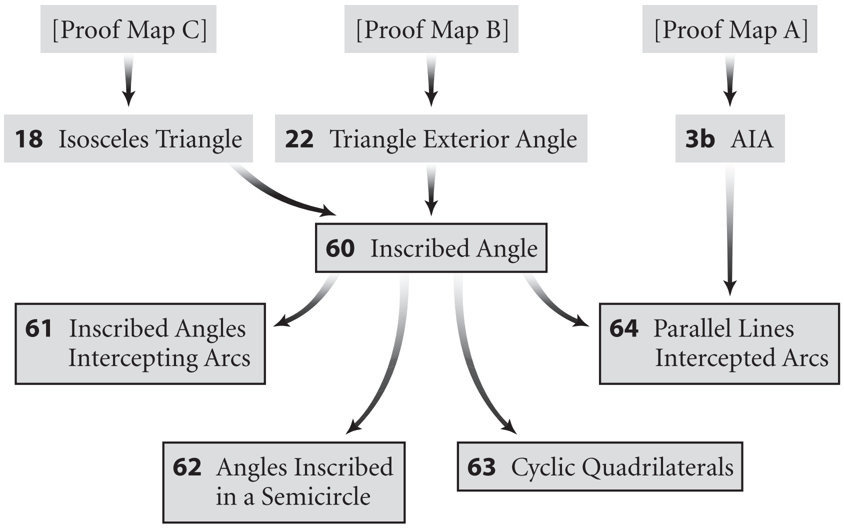
Proof Map G



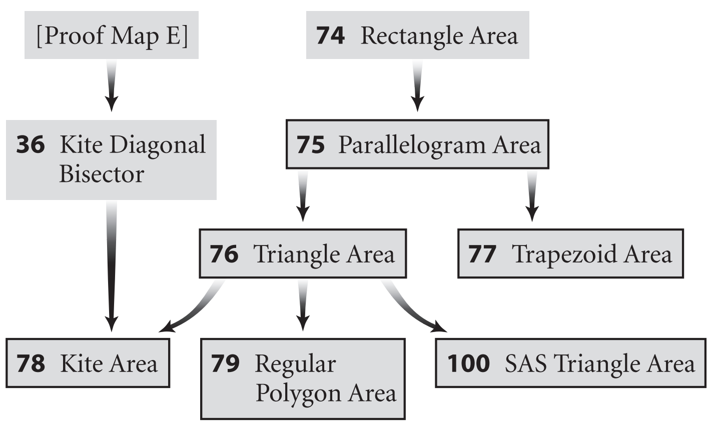
Proof Map H



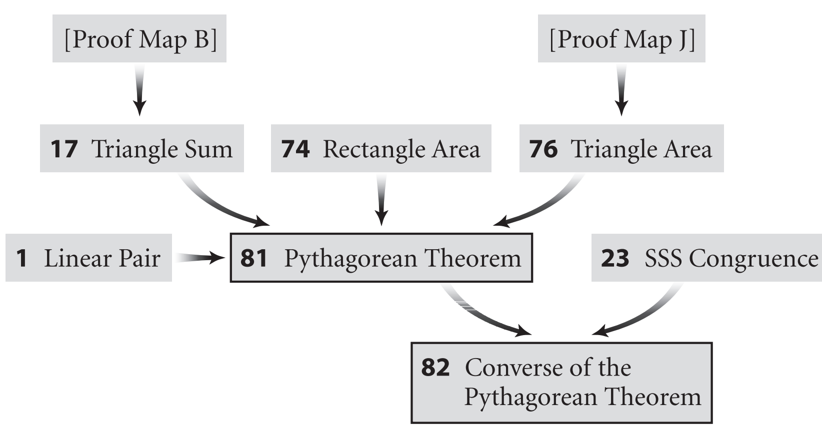
Proof Map I



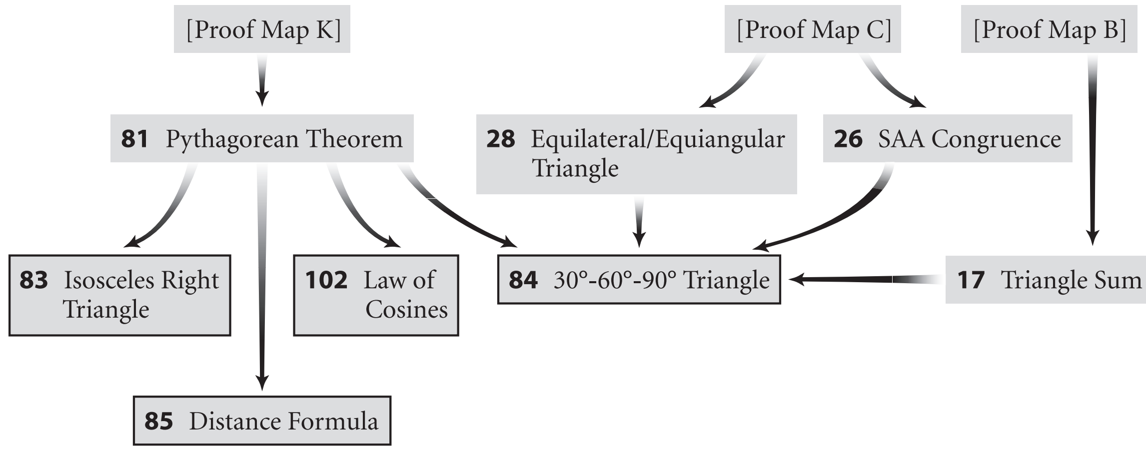
Proof Map J



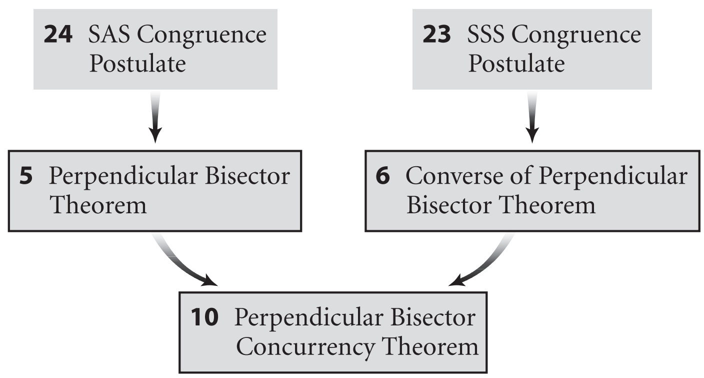
Proof Map K



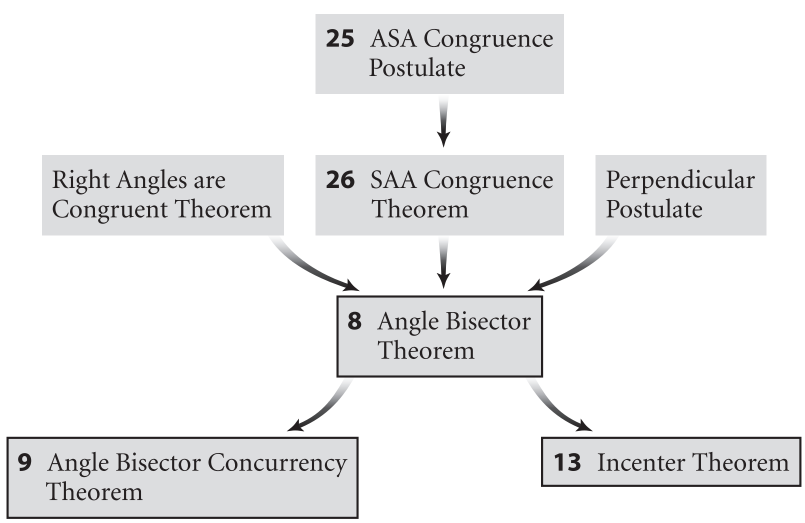
Proof Map L



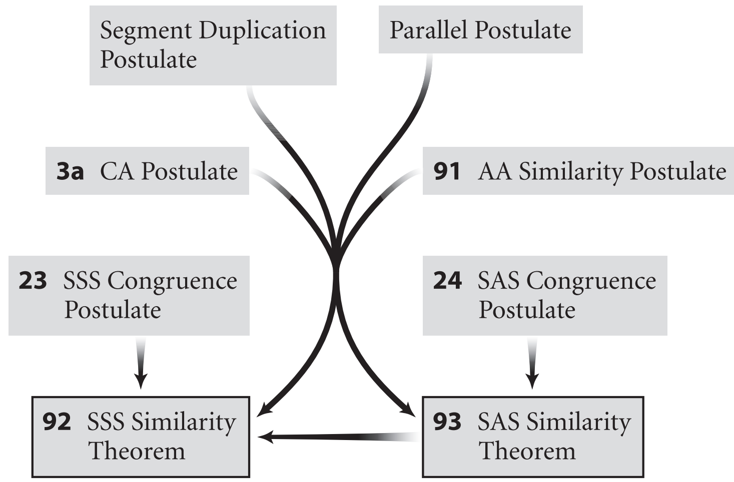
Proof Map M



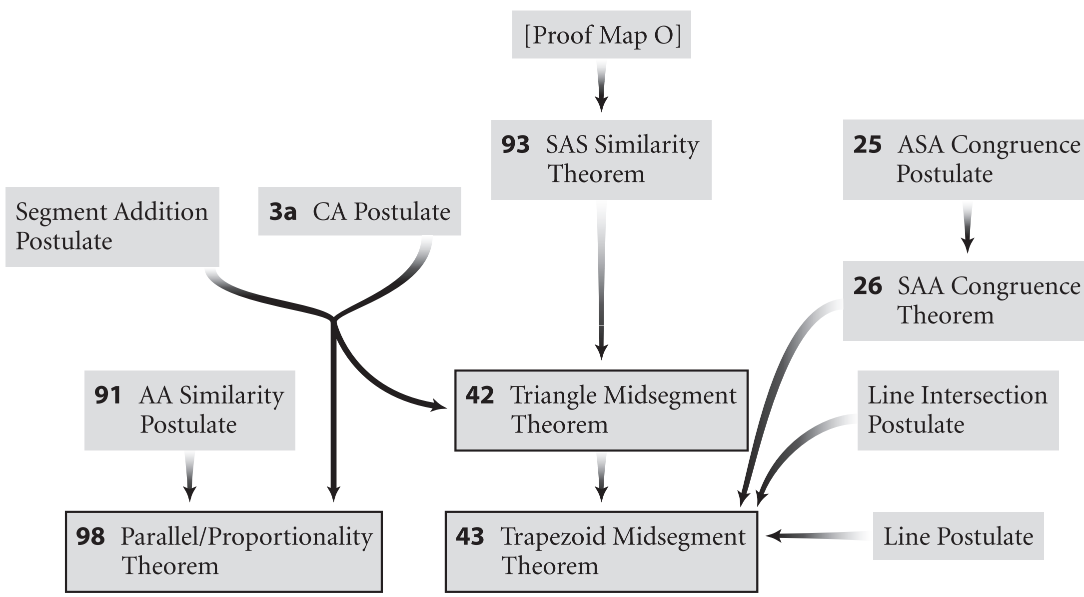
Proof Map N



Proof Map O



Proof Map P



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